

The Ext group in the categories of topological abelian groups and topological vector spaces

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March 8, 2016

Abstract

This paper is devoted to the study of the group $\text{Ext}(G, H)$ of all extensions of topological abelian groups $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$ and the group $\text{Ext}_{\text{TVS}}(Z, Y)$ of all extensions of topological vector spaces $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$. We focus on their behaviour under taking products, countable coproducts, dense subgroups and open subgroups. Finally, we apply the obtained properties to formulate in a more general setting some known results in the category of locally compact abelian groups and to determine conditions in which *being a topological vector space* is a three space property.

Math. Subj. Class. (2010): 54H11, 22B05, 20K35, and 57N17.

1 Introduction

An extension of an abelian group G by an abelian group H is an abelian group $X \geq H$ such that $X/H \cong G$. In the terminology of homological algebra, an extension of abelian groups is a short exact sequence $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$. The set of all extensions of the previous form turns out to be an abelian group and is one of the objects of study of homological algebra.

In [16] Moskowitz studied for the first time the homological algebra in the class of locally compact abelian groups, which we will denote by \mathcal{L} . Among other things he found the injectivities and projectivities of this class. Following his steps, Fulp and Griffith introduced in [12] the group $\text{Ext}(G, H)$ of all extensions $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$ in \mathcal{L} . More recently Sahleh and Alijani found several cases in which the group $\text{Ext}(G, H)$ is trivial under various algebraic and topological conditions imposed on G and H ([19], [20]).

In the framework of functional analysis Kalton, Peck and Roberts provided in [14] the first extensive study of the splitting of extensions in the class of complete metrizable

topological vector spaces. Several problems involving extensions of different types of topological vector spaces were also considered by Domański ([8], [9] and [10]). Latterly Castillo and Simões ([6]) studied the limit properties of the functor Ext in the category of complete locally bounded topological vector spaces.

Our purpose is to investigate the group $\text{Ext}(G, H)$ of all extensions $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$ in the class of topological abelian groups and the group $\text{Ext}_{\text{TVS}}(Y, Z)$ of all extensions $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$ in the class of topological vector spaces. In sections 3 and 4 we will study the properties of Ext and Ext_{TVS} when we take products, countable coproducts, dense subgroups and open subgroups. We will apply these properties to the class \mathcal{L} to show that several theorems proven in [12], [19] and [20] can be formulated in a more general context.

In the fifth section we will use the techniques of section 3 to discuss the following problem: Given two topological vector spaces Y and Z and an extension of topological abelian groups $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$, find conditions in which we can define in X a compatible topological vector space structure in such a way that $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$ becomes an extension of topological vector spaces. This problem was studied by Cattaneo (see [7]) and by Cabello (see [5]) in several situations in which the completeness of Z is required. We will show that their results can be proved without making use of the completeness of Z .

2 Preliminaries

All the topological groups will be Hausdorff. Since we will deal only with abelian groups we will use additive notation.

Given a topological abelian group G and a point $x \in G$ we will use $\mathcal{N}_x(G)$ to denote a system of neighbourhoods of x in G .

We will denote by ω the natural numbers, by \mathbb{R} the set of real numbers, by \mathbb{P} the prime numbers, by \mathbb{Q}_p the p -adic numbers and by \mathbb{Z}_p the p -adic integers. We will use Greek letters to denote ordinal numbers which will be used as index sets.

Given a topological abelian group G , we will call its completion ϱG , analogously, if $f : G \rightarrow H$ is a continuous homomorphism of topological groups, we will call $\varrho f : \varrho G \rightarrow \varrho H$ its completion. A topological abelian group is *precompact* if its completion is a compact group and it is locally precompact if its completion is locally compact ([17, 9.13]).

A topological abelian group G is called *almost metrizable* if there exist a compact subset $K \subseteq G$ with a countable system of neighbourhoods (see [1, Section 4.3]). Metrizable topological groups are almost metrizable. A topological abelian group is Čech-complete if and only if it is almost metrizable and complete ([1, Theorem 4.3.15]). Every locally compact abelian group is Čech-complete (see [17, Lemma 13.13]).

Given $\{G_\alpha : \alpha < \kappa\}$ a family of topological abelian groups the *coproduct topology* on the direct sum $\bigoplus_{\alpha < \kappa} G_\alpha$ is the final group topology with respect to the natural inclusions $G_\gamma \hookrightarrow \bigoplus_{\alpha < \kappa} G_\alpha$. If $\kappa = \omega$ the box topology and the coproduct topology coincide.

We will work with topological vector spaces over the field \mathbb{R} . A topological vector space (shortly t.v.s.) is called Fréchet if it is metrizable, complete and locally convex.

A short exact sequence of topological groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ will be called a *extension of topological groups* (shortly *extension*) when both ι and π are continuous and open onto their images. Two extensions of topological abelian groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ and $E' : 0 \rightarrow H \xrightarrow{\iota'} X' \xrightarrow{\pi'} G \rightarrow 0$ are said to be equivalent if there exists a continuous homomorphism $T : X \rightarrow X'$ making the following diagram commutative

$$\begin{array}{ccccccc} E : & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} G \longrightarrow 0 \\ & & & \parallel & & \downarrow T & \parallel \\ E' : & 0 & \longrightarrow & H & \xrightarrow{\iota'} & X' & \xrightarrow{\pi'} G \longrightarrow 0 \end{array}$$

(i. e. $T \circ \iota = \iota'$, $\pi' \circ T = \pi$). It is known that if such a T exists, it must actually be a topological isomorphism (the proof of this fact is the same as [9, Lemma A]).

An extension of topological abelian groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ *splits* if and only if it is equivalent to the trivial extension $E_0 : 0 \rightarrow H \xrightarrow{\iota_0} H \times G \xrightarrow{\pi_0} G \rightarrow 0$ where ι_0, π_0 are the canonical maps and $H \times G$ is endowed with the product topology. Note that the extension of topological groups E splits if and only if $\iota(H)$ splits as a subgroup of X .

If X, Y and Z are topological vector spaces, a sequence $E : 0 \rightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \rightarrow 0$ is called an *extension of topological vector spaces* if it is an extension of topological abelian groups and the maps ι and π are also linear. An extension of topological vector spaces splits if and only if it splits as an extension of topological abelian groups.

The following known characterization is essential when dealing with extensions of topological groups and topological vector spaces.

Proposition 1. *Let $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ be an extension of topological groups (resp. t.v.s.). The following are equivalent:*

- (i) *E splits.*
- (ii) *There exists a right inverse for π , i.e. a continuous homomorphism (linear mapping) $S : G \rightarrow X$ with $\pi \circ S = \text{id}_G$.*
- (iii) *There exists a left inverse for ι , i.e. a continuous homomorphism (linear mapping) $P : X \rightarrow H$ with $P \circ \iota = \text{id}_H$.*

In the following lemmas we introduce the *push-out* and *pull-back* extensions in the categories of topological abelian groups and topological vector spaces. We will not prove these results because the argument is essentially the same as the one used in abstract abelian groups, with the obvious replacements. For more details see [15, Lemmas 1.2, 1.4 and Theorem 2.1 of chapter III].

Lemma 2. *Let $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ be an extension of topological abelian groups (resp. t.v.s.), let Y be a topological abelian group (t.v.s.), and let $t : Y \rightarrow G$ be a*

continuous homomorphism (linear mapping). The diagram

$$E : \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{\pi} & G \longrightarrow 0 \\ & & & & & & \uparrow t \\ & & & & & & Y \end{array} \quad (1)$$

can be completed to a diagram of the form

$$\begin{array}{ccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & \parallel & & \uparrow s & & \uparrow t & & \\ Et: & 0 & \longrightarrow & H & \xrightarrow{I} & PB & \xrightarrow{r} & Y & \longrightarrow & 0 \end{array}$$

where PB , r and s form the pull-back of π and t in the category of topological abelian groups (t.v.s.). The bottom sequence Et is an extension of topological abelian groups (t.v.s.) which will be called the pull-back extension. Furthermore if $\mathcal{E} : 0 \rightarrow H \xrightarrow{I'} X' \xrightarrow{r'} G \rightarrow 0$ is another extension of topological abelian groups (t.v.s.) that completes the diagram (1) in the same way, then \mathcal{E} and Et must be equivalent.

From now on we will use the notation Et to denote the *pull-back extension* of an extension E with respect to a continuous homomorphism t .

Lemma 3. *Let $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ be an extension of topological abelian groups (resp. t.v.s), let Y be a topological abelian group (resp. t.v.s.), and let $t : H \rightarrow Y$ be a continuous homomorphism (linear mapping). The diagram*

$$E: \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} & G \longrightarrow 0 \\ & & \downarrow t & & & & \\ & & Y & & & & \end{array} \quad (2)$$

can be completed to a diagram of the form

$$\begin{array}{ccccccccc} E: & 0 & \longrightarrow & H & \xrightarrow{u} & X & \xrightarrow{\pi} & G & \longrightarrow 0 \\ & & & \downarrow t & & \downarrow s & & \parallel & \\ tE: & 0 & \longrightarrow & Y & \xrightarrow{r} & PO & \xrightarrow{p} & G & \longrightarrow 0 \end{array}$$

where PO , r and s form the push-out of ι and t in the category of topological abelian groups (t.v.s.). The bottom sequence tE is an extension of topological abelian groups (t.v.s.) which will be called the push-out extension. Furthermore if $\mathcal{E} : 0 \rightarrow H \xrightarrow{r'} Y \xrightarrow{p'} G \rightarrow 0$ is another extension of topological abelian groups (t.v.s.) that completes the diagram (2) in the same way, then \mathcal{E} and tE must be equivalent.

From now on we will use the notation tE to denote the *push-out extension* of an extension E with respect to a continuous homomorphism t .

Given two topological groups G and H , the set of equivalence classes of extensions of topological groups of the form $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$ will be denoted by $\text{Ext}(G, H)$. We will write $\text{Ext}(G, H) = 0$ when every extension $0 \rightarrow H \rightarrow X \rightarrow G \rightarrow 0$ splits.

Let $E_1 : 0 \rightarrow H \xrightarrow{i_1} X_1 \xrightarrow{\pi_1} G \rightarrow 0$ and $E_2 : 0 \rightarrow H \xrightarrow{i_2} X_2 \xrightarrow{\pi_2} G \rightarrow 0$ be two extensions of topological abelian groups. Consider the canonical mappings $\nabla_H : H \times H \rightarrow H$ $[(h, h') \mapsto h + h']$ and $\Delta_G : G \rightarrow G \times G$ $[g \mapsto (g, g)]$ and the extension of topological abelian groups $E_1 \times E_2 : 0 \rightarrow H \times H \xrightarrow{i_1 \times i_2} X_1 \times X_2 \xrightarrow{\pi_1 \times \pi_2} G \times G \rightarrow 0$. We will define the sum $E_1 + E_2$ as the extension $\nabla_H((E_1 \times E_2)\Delta_G)$. This operation is called the Baer sum. The set $\text{Ext}(G, H)$ with the operation induced by Baer sum in the equivalence classes of extensions of topological abelian groups is an abelian group (the proof is essentially the same as [15, Theorem 2.1 of Chapter III]).

Given two topological vector spaces Y, Z we will call $\text{Ext}_{\text{TVS}}(Z, Y)$ to the set of equivalence classes of extensions of topological vector spaces of the form $0 \rightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \rightarrow 0$. Taking the operation induced by the Baer sum defined in the same way, we can endow $\text{Ext}_{\text{TVS}}(Z, Y)$ with an structure of abelian group.

We will say that a complete metrizable topological vector space Z is a \mathcal{K} -space if $\text{Ext}_{\text{TVS}}(Z, \mathbb{R}) = 0$ (see [14, Chapter 5]).

The following lemma will be very useful. The proof is the same as the one in [15, Theorem 2.1 of Chapter III].

Lemma 4. *Let G, H, Y_1 and Y_2 be topological abelian groups. Suppose that $t_1 : Y_1 \rightarrow G$ and $t_2 : H \rightarrow Y_2$ are continuous homomorphisms. Then the following maps are homomorphisms of abelian groups*

$$\begin{array}{ccccc} \text{Ext}(G, H) & \longrightarrow & \text{Ext}(Y_1, H) & & \text{Ext}(G, H) & \longrightarrow & \text{Ext}(G, Y_2) \\ [E] & \longmapsto & [Et_1] & & [E] & \longmapsto & [t_2E] \end{array}$$

The analogous statement for topological vector spaces and continuous linear mappings is also true.

3 Properties of the Ext group with respect to open subgroups and dense subgroups

Lemma 5. *([3, Proposition 3.10]) Let $E : 0 \rightarrow H \xrightarrow{i} X \xrightarrow{\pi} G \rightarrow 0$ be an extension of topological abelian groups. Suppose that the completion of H is a Čech-complete group. Then the sequence $\varrho E : 0 \rightarrow \varrho H \xrightarrow{\varrho i} \varrho X \xrightarrow{\varrho \pi} \varrho G \rightarrow 0$ is an extension of topological abelian groups.*

Theorem 6. *Let G, H be topological Abelian groups. If H is Čech-complete then $\text{Ext}(G, H) \cong \text{Ext}(\varrho G, H)$.*

Proof. Consider the canonical inclusion $\mathcal{I} : G \rightarrow \varrho G$. According to Lemma 4 the map

$$\begin{array}{ccc} \phi : \text{Ext}(\varrho G, H) & \longrightarrow & \text{Ext}(G, H) \\ [E] & \longmapsto & [E\mathcal{I}] \end{array}$$

is a homomorphism of abelian groups. Let us see that ϕ is an isomorphism.

To prove that ϕ is injective, pick an extension of topological abelian groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} \varrho G \rightarrow 0$ and suppose that $E\mathcal{I}$ splits. An easy verification shows that the sequence $E' : 0 \rightarrow H \xrightarrow{\iota} \pi^{-1}(G) \xrightarrow{\pi|_{\pi^{-1}(G)}} G \rightarrow 0$ is an extension of topological abelian groups. Furthermore the following diagram is commutative

$$\begin{array}{ccccccc} E : & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} \varrho G \longrightarrow 0 \\ & & & \parallel & & \uparrow \mathcal{I} & \\ E' : & 0 & \longrightarrow & H & \xrightarrow{\iota} & \pi^{-1}(G) & \xrightarrow{\pi|_{\pi^{-1}(G)}} G \longrightarrow 0 \end{array}$$

According to Lemma 2, E' must be equivalent to $E\mathcal{I}$. Then E' splits and applying Proposition 1 we find a continuous homomorphism $P : \pi^{-1}(G) \rightarrow H$ such that $P \circ \iota = \text{Id}_H$. Since G is dense in ϱG , it is clear that $\pi^{-1}(G)$ is dense in X . Call $R : X \rightarrow H$ the canonical extension of P to X . R is a continuous homomorphism satisfying $R \circ \iota = \text{Id}_H$, hence by Proposition 1, E splits.

To check that ϕ is onto, choose an extension of topological abelian groups $\mathcal{E} : 0 \rightarrow H \xrightarrow{I} Y \xrightarrow{p} G \rightarrow 0$. By Lemma 5 the sequence $\varrho\mathcal{E} : 0 \rightarrow H \xrightarrow{\varrho I} \varrho Y \xrightarrow{\varrho p} \varrho G \rightarrow 0$ is an extension of topological abelian groups. The following diagram is commutative

$$\begin{array}{ccccccc} \varrho\mathcal{E} : & 0 & \longrightarrow & H & \xrightarrow{\varrho I} & \varrho Y & \xrightarrow{\varrho p} \varrho G \longrightarrow 0 \\ & & & \parallel & & \uparrow \mathcal{I} & \\ \mathcal{E} : & 0 & \longrightarrow & H & \xrightarrow{I} & Y & \xrightarrow{p} G \longrightarrow 0 \end{array}$$

In virtue of Lemma 2, E must be equivalent to $(\varrho\mathcal{E})\mathcal{I}$ hence $\phi([\varrho\mathcal{E}]) = [\mathcal{E}]$ \square

Theorem 7. *Let G be a topological abelian group and let H be a Čech-complete topological abelian group. Suppose that D is a dense subgroup of G . Then $\text{Ext}(G, H) \cong \text{Ext}(D, H)$.*

Proof. If D is dense in then G $\varrho D = \varrho G$. By Theorem 6

$$\text{Ext}(D, H) \cong \text{Ext}(\varrho D, H) = \text{Ext}(\varrho G, H) \cong \text{Ext}(G, H).$$

\square

In theorems 3.5 and 3.6 of [12] the authors study situations in which $\text{Ext}(G, X) = 0$ for a fixed $G \in \mathcal{L}$ and X varying in a subclass of \mathcal{L} . The following corollary is an application of Theorem 6 on these results.

Corollary 8. *Let G be a locally precompact abelian group.*

- (i) *$\text{Ext}(G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$ if and only if $\varrho G = (\bigoplus_{\alpha < \kappa} \mathbb{Z}) \times \mathbb{R}^n$ for some $n \in \omega$ and an arbitrary ordinal number κ .*

(ii) $\text{Ext}(G, X) = 0$ for all connected $X \in \mathcal{L}$ if and only if $\rho G = \mathbb{R}^n \times G'$ where $n \in \mathbb{Z}^+$ and G' contains a compact open subgroup having a co-torsion dual.

Proof. (i) Suppose that a locally precompact abelian group G has the property that $\text{Ext}(G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. Since every group in \mathcal{L} is Čech-complete, by Theorem 6 $\text{Ext}(\rho G, X) = \text{Ext}(G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. By [12, Theorem 3.5], $\rho G = (\bigoplus_{\alpha} \mathbb{Z}) \times \mathbb{R}^n$ for some $n \in \omega$ and α an ordinal number. Conversely if $\rho G = (\bigoplus_{\alpha} \mathbb{Z}) \times \mathbb{R}^n$ by [12, Theorem 3.5], $\text{Ext}(\rho G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$. Using Theorem 6 we see that $\text{Ext}(G, X) = \text{Ext}(\rho G, X) = 0$ for all totally disconnected $X \in \mathcal{L}$.

(ii) Apply the same argument to [12, Theorem 3.6]. □

In the following examples, we compute several Ext groups:

Example 9. Consider \mathbb{Q}_d the rational numbers with the discrete topology and D any dense subgroup of \mathbb{T} . Denote by Σ_a the a -adic solenoid that satisfies $\mathbb{Q}_d^{\wedge} \cong \Sigma_a$ (see [13, 25.4]). According to [11, Exercise 51.7], $\text{Ext}(\mathbb{Q}_d, \mathbb{Z}) \cong \mathbb{Q}^{\omega}$. By [12, Theorem 2.12], $\text{Ext}(\mathbb{Q}_d, \mathbb{Z}) \cong \text{Ext}(\mathbb{Z}^{\wedge}, \mathbb{Q}_d^{\wedge}) \cong \text{Ext}(\mathbb{T}, \Sigma_a)$. Finally, in virtue of Theorem 7, $\text{Ext}(D, \Sigma_a) \cong \mathbb{Q}^{\omega}$.

Example 10. Let G be a product of locally precompact abelian torsion groups. Consider $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \rightarrow 0$ the canonical extension of \mathbb{T} by \mathbb{Z} . It is easy to see that [12, Theorem 2.14] is also valid for topological abelian groups outside the class \mathcal{L} , hence we can consider the classical Hom-Ext exact sequence of abelian groups in this context

$$\begin{aligned} 0 \rightarrow \text{CHom}(G, \mathbb{Z}) \rightarrow \text{CHom}(G, \mathbb{R}) \rightarrow \text{CHom}(G, \mathbb{T}) \\ \rightarrow \text{Ext}(G, \mathbb{Z}) \rightarrow \text{Ext}(G, \mathbb{R}) \end{aligned}$$

$\text{CHom}(G, \mathbb{R}) = 0$ because G is a torsion group. According to [3, Corollary 3.15], since G is a product of locally precompact abelian groups, $\text{Ext}(G, \mathbb{R}) = 0$. Then $\text{Ext}(G, \mathbb{Z}) \cong \text{CHom}(G, \mathbb{T})$. If we only ask G to be a topological abelian torsion group, the same argument shows that $\text{Ext}(G, \mathbb{Z})$ contains $\text{CHom}(G, \mathbb{T})$ as a subgroup.

Theorem 11. *Let Y, Z be topological vector spaces. If Y is complete and metrizable then $\text{Ext}_{\text{TVS}}(Z, Y) \cong \text{Ext}_{\text{TVS}}(\rho Z, Y)$.*

Proof. Since Y is complete and metrizable, it is Čech-complete and we are in the conditions of Lemma 5. Notice that if we use Lemma 5 to complete an extension of topological vector spaces we obtain an extension of topological vector spaces. Having this in mind we can repeat the proof of Theorem 6 in this context. □

The following result generalizes [9, Proposition 4.2]:

Theorem 12. *Let Z be a topological vector space and let Y be a complete metrizable topological vector space. If D is a dense subspace of Z then $\text{Ext}_{\text{TVS}}(Z, Y) \cong \text{Ext}_{\text{TVS}}(D, Y)$*

Proof. Proceed as in Theorem 7 using Theorem 11 instead of Theorem 6. \square

Lemma 13. *Let A be an open subgroup of a topological group G and suppose that an extension of topological abelian groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} A \rightarrow 0$ splits algebraically. Then there exists a group topology τ on $H \times G$ and an embedding $f : X \rightarrow (H \times G, \tau)$ making commutative the diagram*

$$\begin{array}{ccccccc} \overline{E} : & 0 & \longrightarrow & H & \xrightarrow{\iota_\tau} & (H \times G, \tau) & \xrightarrow{\pi_\tau} G \longrightarrow 0 \\ & & & \parallel & & \uparrow f & \uparrow \\ E & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} A \longrightarrow 0 \end{array}$$

where ι_τ and π_τ are the canonical mappings and \overline{E} is an extension of topological abelian groups.

Proof. Since E splits algebraically there exists a group topology τ' on $H \times A$ such that E is equivalent to the extension of topological abelian groups $\mathcal{E} : 0 \rightarrow H \xrightarrow{\iota_{\tau'}} (H \times A, \tau') \xrightarrow{\pi_{\tau'}} A \rightarrow 0$ where $\iota_{\tau'}$ and $\pi_{\tau'}$ are respectively the canonical inclusion and the canonical projection. Call T the topological isomorphism making the following diagram commutative

$$\begin{array}{ccccccc} \mathcal{E} : & 0 & \longrightarrow & H & \xrightarrow{\iota_{\tau'}} & (H \times A, \tau') & \xrightarrow{\pi_{\tau'}} A \longrightarrow 0 \\ & & & \parallel & & \uparrow T & \parallel \\ E : & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} A \longrightarrow 0 \end{array} \quad (3)$$

Now, consider on $H \times G$ the group topology τ obtained by declaring $(H \times A, \tau')$ an open subgroup. An easy verification shows that if we call $\iota_\tau : H \rightarrow (H \times G, \tau)$ the canonical inclusion and $\pi_\tau : (H \times G, \tau) \rightarrow G$ the canonical projection, the sequence $\overline{E} : 0 \rightarrow H \xrightarrow{\iota_\tau} (H \times G, \tau) \xrightarrow{\pi_\tau} G \rightarrow 0$ is an extension of topological abelian groups. Combining the commutative diagram

$$\begin{array}{ccccccc} \overline{E} : & 0 & \longrightarrow & H & \xrightarrow{\iota_\tau} & (H \times G, \tau) & \xrightarrow{\pi_\tau} G \longrightarrow 0 \\ & & & \parallel & & \uparrow & \uparrow \\ \mathcal{E} & 0 & \longrightarrow & H & \xrightarrow{\iota_{\tau'}} & (H \times A, \tau') & \xrightarrow{\pi_{\tau'}} A \longrightarrow 0 \end{array}$$

with (3) and defining f as the composition of T and the inclusion $(H \times A, \tau') \hookrightarrow (H \times G, \tau)$, we complete the proof. \square

Theorem 14. *Let G, H be topological abelian groups. Suppose that H is divisible and that A is an open subgroup of G . Then $\text{Ext}(G, H) \cong \text{Ext}(A, H)$.*

Proof. We will use the same strategy as in Theorem 6. Consider the canonical inclusion $\mathcal{I} : A \rightarrow G$. According to Lemma 4 the map

$$\begin{array}{ccc} \phi : \text{Ext}(G, H) & \longrightarrow & \text{Ext}(A, H) \\ [E] & \longmapsto & [E\mathcal{I}] \end{array}$$

is a homomorphism of abelian groups. Let us see that ϕ is an isomorphism.

To prove that ϕ is injective, pick an extension of topological abelian groups $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$ and suppose that $E\mathcal{I}$ splits. Since $\iota(H) \leq \pi^{-1}(G)$, the sequence $E' : 0 \rightarrow H \xrightarrow{\iota} \pi^{-1}(A) \xrightarrow{\pi|_{\pi^{-1}(A)}} G \rightarrow 0$ is exact. The mapping $\pi|_{\pi^{-1}(A)}$ is open hence E' is a topological extension. Furthermore the following diagram is commutative:

$$\begin{array}{ccccccc} E : & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} G \longrightarrow 0 \\ & & & \parallel & & \uparrow & \uparrow \mathcal{I} \\ E' : & 0 & \longrightarrow & H & \xrightarrow{\iota} & \pi^{-1}(A) & \xrightarrow{\pi|_{\pi^{-1}(A)}} A \longrightarrow 0 \end{array}$$

According to Lemma 2, E' must be equivalent to $E\mathcal{I}$. Then E' splits and applying Proposition 1 we find a continuous homomorphism $P : \pi^{-1}(A) \rightarrow H$ such that $P \circ \iota = \text{Id}_H$. Since H is divisible we can extend the homomorphism P to a homomorphism $R : X \rightarrow H$. Since $\pi^{-1}(A)$ is open in X and $R|_{\pi^{-1}(A)} = P$, R is a continuous homomorphism. As $\iota(H) \leq \pi^{-1}(A)$, R satisfies that $R \circ \iota = P \circ \iota = \text{Id}_H$ and by Proposition 1, E splits.

To check that ϕ is onto, choose an extension of topological abelian groups $E : 0 \rightarrow H \xrightarrow{I} Y \xrightarrow{p} A \rightarrow 0$. From Lemma 13 we know that there exists a group topology τ on $H \times G$ and a commutative diagram

$$\begin{array}{ccccccc} \overline{E} : & 0 & \longrightarrow & H & \xrightarrow{\iota_\tau} (H \times G, \tau) & \xrightarrow{\pi_\tau} G & \longrightarrow 0 \\ & & & \parallel & & \uparrow \mathcal{I} & \\ E & 0 & \longrightarrow & H & \xrightarrow{I} Y & \xrightarrow{p} A & \longrightarrow 0 \end{array}$$

where ι_τ and π_τ are the canonical mappings and \overline{E} is an extension of topological abelian groups. In virtue of Lemma 2, E must be equivalent to $(\overline{E})\mathcal{I}$, concluding that $\phi([E]) = [(\overline{E})\mathcal{I}] = [E]$. □

Remark 15. In [3, Proposition 3.11] the authors prove that for every Čech-complete topological abelian group H , if $\text{Ext}(\varrho G, H) = 0$ then $\text{Ext}(G, H) = 0$. Theorem 6 is a generalization of this fact. In [2, Corollary 14] it is proven that if A is an open subgroup of a topological abelian group G and $\text{Ext}(A, \mathbb{T}) = 0$ then $\text{Ext}(G, \mathbb{T}) = 0$. Theorem 14 generalizes this result.

4 Properties of the Ext group with respect to products and coproducts

Theorem 16. (i) Let G be a topological abelian group and let $\{H_\alpha : \alpha < \kappa\}$ be a family of topological abelian groups. Then $\text{Ext}(G, \prod_{\alpha < \kappa} H_\alpha) \cong \prod_{\alpha < \kappa} \text{Ext}(G, H_\alpha)$.

(ii) Let Z be a topological vector space and let $\{Y_\alpha : \alpha < \kappa\}$ be a family of topological vector spaces. Then $\text{Ext}_{\text{TVS}}(Z, \prod_{\alpha < \kappa} Y_\alpha) \cong \prod_{\alpha < \kappa} \text{Ext}_{\text{TVS}}(Z, Y_\alpha)$.

Proof. (i). Consider for every $\beta < \kappa$ the canonical projection $p_\beta : \prod_{\alpha < \kappa} H_\alpha \rightarrow H_\beta$. Given an extension of topological abelian groups $E : 0 \rightarrow \prod_{\alpha < \kappa} H_\alpha \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$, take the push-out extension $p_\beta E : 0 \rightarrow H_\beta \xrightarrow{r_\beta} PO_\beta \xrightarrow{P_\beta} G \rightarrow 0$ with respect to p_β and consider the commutative diagram (4) as in Lemma 3.

$$\begin{array}{ccccccc} E : & 0 & \longrightarrow & \prod_{\alpha < \kappa} H_\alpha & \xrightarrow{\iota} & X & \xrightarrow{\pi} G \longrightarrow 0 \\ & & & \downarrow p_\beta & & \downarrow s_\beta & \parallel \\ p_\beta E : & 0 & \longrightarrow & H_\beta & \xrightarrow{r_\beta} & PO_\beta & \xrightarrow{P_\beta} G \longrightarrow 0 \end{array} \quad (4)$$

In virtue of Lemma 4 the map

$$\begin{array}{ccc} \phi : \text{Ext}(G, \prod_{\alpha < \kappa} H_\alpha) & \longrightarrow & \prod_{\alpha < \kappa} \text{Ext}(G, H_\alpha) \\ [E] & \longmapsto & ([p_\alpha E])_{\alpha < \kappa} \end{array}$$

is a homomorphism of abelian groups.

Let us check that ϕ is injective. Take an extension of topological groups $E : 0 \rightarrow \prod_{\alpha < \kappa} H_\alpha \xrightarrow{\iota} X \xrightarrow{\pi} G \rightarrow 0$, and suppose that $p_\beta E : 0 \rightarrow H_\beta \xrightarrow{r_\beta} PO_\beta \xrightarrow{P_\beta} G \rightarrow 0$ splits for every $\beta < \kappa$. By Proposition 1 for every $\alpha < \kappa$ there exists a continuous homomorphism $t_\alpha : PO_\alpha \rightarrow H_\alpha$ with $t_\alpha \circ r_\alpha = \text{Id}_{H_\alpha}$. Define the continuous homomorphism

$$\begin{array}{ccc} T : X & \longrightarrow & \prod_{\alpha < \kappa} H_\alpha \\ x & \longmapsto & (t_\alpha(s_\alpha(x)))_{\alpha < \kappa} \end{array}$$

By the commutativity of (4), $T \circ \iota = \text{Id}_{\prod_{\alpha < \kappa} H_\alpha}$, thus E splits.

To see that ϕ is onto pick a family of extensions $\{E_\alpha : 0 \rightarrow H_\alpha \xrightarrow{\iota_\alpha} X_\alpha \xrightarrow{\pi_\alpha} G \rightarrow 0 : \alpha < \kappa\}$. Consider the extension $\mathcal{E} : 0 \rightarrow \prod_{\alpha < \kappa} H_\alpha \xrightarrow{\mathcal{I}} \mathcal{B} \xrightarrow{\mathcal{P}} G \rightarrow 0$, where

$$\mathcal{B} = \left\{ ((x_\alpha)_{\alpha < \kappa}, g) \in \prod_{\alpha < \kappa} X_\alpha \times G : \pi_\alpha(x_\alpha) = g \ \forall \alpha < \kappa \right\},$$

$\mathcal{I}((h_\alpha)_{\alpha < \kappa}) = ((\iota_\alpha(h_\alpha))_{\alpha < \kappa}, 0)$ and $\mathcal{P}((x_\alpha)_{\alpha < \kappa}, g) = g$. It is easy to check that \mathcal{E} is an extension of topological abelian groups. Define for each $\beta < \kappa$ the continuous homomorphism

$$\begin{array}{ccc} \mathcal{P}_\beta : \mathcal{B} & \longrightarrow & X_\beta \\ ((x_\alpha)_{\alpha < \kappa}, g) & \longmapsto & x_\beta \end{array}$$

The following diagram is commutative for every $\beta < \kappa$.

$$\begin{array}{ccccccc}
\mathcal{E} : & 0 & \longrightarrow & \prod_{\alpha < \kappa} H_\alpha & \xrightarrow{\mathcal{I}} & \mathcal{B} & \xrightarrow{\mathcal{P}} G \longrightarrow 0 \\
& & & \downarrow p_\beta & & \downarrow \mathcal{P}_\beta & \parallel \\
E_\beta : & 0 & \longrightarrow & H_\beta & \xrightarrow{\iota_\beta} & X_\beta & \xrightarrow{\pi_\beta} G \longrightarrow 0
\end{array}$$

Consequently, by Lemma 3, E_β must be equivalent to the push-out sequence $p_\beta \mathcal{E}$. Hence $\varphi([\mathcal{E}]) = ([E_\alpha])_{\alpha < \kappa}$, which concludes the proof.

(ii). All the steps in the previous part remain valid in the category of topological vector spaces so we can proceed in the same way. \square

Remark 17. In [3, Proposition 3.12] it is proved that given a family of topological abelian groups $\{H_\alpha : \alpha < \kappa\}$ and a topological group G , $\text{Ext}(G, H_\alpha) = 0$ for every $\alpha < \kappa$ if and only if $\prod_{\alpha < \kappa} \text{Ext}(G, H_\alpha) = 0$. This result is a particular case of Theorem 16.

Corollary 18. *Let G be a locally precompact abelian group. Then $\text{Ext}(G, \prod_{p \in \mathbb{P}} \mathbb{Q}_p^{\alpha_p})$ is a divisible, torsion-free group for every collection of ordinal numbers $\alpha_p, p \in \mathbb{P}$.*

Proof. According to Theorem 16

$$\text{Ext}(G, \prod_{p \in \mathbb{P}} \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p \in \mathbb{P}} \text{Ext}(G, \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p \in \mathbb{P}} \text{Ext}(G, \mathbb{Q}_p)^{\alpha_p}.$$

Since \mathbb{Q}_p is Čech-complete, by Theorem 6 $\text{Ext}(G, \mathbb{Q}_p) \cong \text{Ext}(\varrho G, \mathbb{Q}_p)$ for every $p \in \mathbb{P}$ and

$$\text{Ext}(G, \prod_{p \in \mathbb{P}} \mathbb{Q}_p^{\alpha_p}) \cong \prod_{p \in \mathbb{P}} \text{Ext}(\varrho G, \mathbb{Q}_p)^{\alpha_p}.$$

$\text{Ext}(\varrho G, \mathbb{Q}_p)$ is divisible and torsion-free by [20, Corollary 1.5], then $\text{Ext}(G, \prod_{p \in \mathbb{P}} \mathbb{Q}_p^{\alpha_p})$ is also divisible and torsion-free. \square

Corollary 19. *Let G be a locally precompact abelian group. $\text{Ext}(G, X) = 0$ for all X product of divisible σ -compact groups in \mathcal{L} if and only if $\varrho G = \mathbb{R}^n \times G'$ where n is a non-negative integer and G' contains a compact open subgroup K such that $K \cong \prod_{\alpha < \alpha_0} \mathbb{Z}/p_\alpha^{\gamma_\alpha} \mathbb{Z} \times \prod_{\beta < \beta_0} \mathbb{Z}_{p_\beta}^{\gamma_\beta}$ where $\alpha_0, \beta_0 \in \omega$, $\{\gamma_\beta : \beta < \beta_0\}$ is a family of arbitrary ordinal numbers and p_α, p_β are prime numbers for all α, β .*

Proof. Suppose that a locally precompact abelian group G has the property that $\text{Ext}(G, X) = 0$ for all X product of divisible σ -compact groups in \mathcal{L} . In particular $\text{Ext}(G, X) = 0$ for all X divisible σ -compact in \mathcal{L} . Since every group in \mathcal{L} is Čech-complete, by Theorem 6, $\text{Ext}(\varrho G, X) = \text{Ext}(G, X) = 0$, for all divisible σ -compact $X \in \mathcal{L}$. According to [19, Theorem 2.7], ϱG has the desired structure.

Conversely if ϱG has the properties described in the statement, in virtue of [19, Theorem 2.7], $\text{Ext}(\varrho G, X) = 0$ for all X divisible σ -compact group \mathcal{L} . By Theorem 6, $\text{Ext}(G, X) = 0$ for every X divisible σ -compact in \mathcal{L} . Finally, from Theorem 16 we conclude that the same is true for every X product of divisible σ -compact groups in \mathcal{L} . \square

Corollary 20. *Let G be a torsion group in \mathcal{L} and let H be product of divisible torsion-free groups in \mathcal{L} . Then $\text{Ext}(G, H) = 0$.*

Proof. Suppose that $H = \prod_{\alpha < \kappa} H_\alpha$ with H_α divisible, torsion-free and $H_\alpha \in \mathcal{L}$ for every $\alpha < \kappa$. Since G is locally compact abelian and torsion we know by [13, 24.21] that G contains an open compact subgroup K . It is clear that K will be a torsion group too. Applying Theorem 16 and Theorem 14

$$\text{Ext}(G, H) \cong \text{Ext}(G, \prod_{\alpha < \kappa} H_\alpha) \cong \prod_{\alpha < \kappa} \text{Ext}(G, H_\alpha) \cong \prod_{\alpha < \kappa} \text{Ext}(K, H_\alpha).$$

Finally by [20, Theorem 1.6] $\text{Ext}(K, H_\alpha) = 0$ for every $\alpha < \kappa$. □

Lemma 21. *Let $\{E_\alpha : 0 \rightarrow H_\alpha \xrightarrow{\iota_\alpha} X_\alpha \xrightarrow{\pi_\alpha} G_\alpha \rightarrow 0 : \alpha < \omega\}$ be a countable family of extensions of topological abelian groups. Consider the coproducts $\bigoplus_{\alpha < \omega} H_\alpha$, $\bigoplus_{\alpha < \omega} X_\alpha$, $\bigoplus_{\alpha < \omega} G_\alpha$ and the natural mappings $\bigoplus_{\alpha < \omega} \iota_\alpha : \bigoplus_{\alpha < \omega} H_\alpha \rightarrow \bigoplus_{\alpha < \omega} X_\alpha$ and $\bigoplus_{\alpha < \omega} \pi_\alpha : \bigoplus_{\alpha < \omega} X_\alpha \rightarrow \bigoplus_{\alpha < \omega} G_\alpha$. The sequence*

$$\bigoplus_{\alpha < \omega} E_\alpha : 0 \longrightarrow \bigoplus_{\alpha < \omega} H_\alpha \xrightarrow{\bigoplus_{\alpha < \omega} \iota_\alpha} \bigoplus_{\alpha < \omega} X_\alpha \xrightarrow{\bigoplus_{\alpha < \omega} \pi_\alpha} \bigoplus_{\alpha < \omega} G_\alpha \longrightarrow 0$$

is an extension of topological abelian groups.

Proof. Straightforward. □

Theorem 22. *Let H be a topological abelian group and let $\bigoplus_{\alpha < \omega} G_\alpha$ a countable co-product of topological abelian groups. Then $\text{Ext}(\bigoplus_{\alpha < \omega} G_\alpha, H) \cong \prod_{\alpha < \omega} \text{Ext}(G_\alpha, H)$.*

Proof. Consider the canonical inclusion $\mathcal{I}_\alpha : G_\alpha \rightarrow \bigoplus_{\alpha < \omega} G_\alpha$ and define

$$\begin{aligned} \phi : \text{Ext}(\bigoplus_{\alpha < \omega} G_\alpha, H) &\longrightarrow \prod_{\alpha < \omega} \text{Ext}(G_\alpha, H) \\ [E] &\longmapsto ([E\mathcal{I}_\alpha])_{\alpha < \omega} \end{aligned}$$

According to Lemma 2, ϕ is a homomorphism of abelian groups. Let us see that ϕ is an isomorphism.

To see that ϕ is injective take $E : 0 \rightarrow H \xrightarrow{\iota} X \xrightarrow{\pi} \bigoplus_{\alpha < \omega} G_\alpha \rightarrow 0$ an extension of topological abelian groups and suppose that $E\mathcal{I}_\beta$ splits for every $\beta < \omega$. Pick $\beta < \omega$.

Take the sequence $E_\beta : 0 \rightarrow H \xrightarrow{\iota} \pi^{-1}(G_\beta) \xrightarrow{\pi|_{\pi^{-1}(G_\beta)}} G_\beta \rightarrow 0$. Since $\iota(H) \subset \pi^{-1}(G_\beta)$, E_β is an exact sequence. Since $\pi|_{\pi^{-1}(G_\beta)}$ is open (see [4, Proposition 2, Chapter 5.1]) it follows that E_β is an extension of topological abelian groups. For every $\beta < \omega$ the following diagram is commutative

$$\begin{array}{ccccccc} E : & 0 & \longrightarrow & H & \xrightarrow{\iota} & X & \xrightarrow{\pi} \bigoplus_{\alpha < \omega} G_\alpha \longrightarrow 0 \\ & & & \parallel & & \uparrow & \uparrow \mathcal{I}_\beta \\ E_\beta : & 0 & \longrightarrow & H & \xrightarrow{\iota} \pi^{-1}(G_\beta) & \xrightarrow{\pi|_{\pi^{-1}(G_\beta)}} G_\beta \longrightarrow 0 \end{array}$$

Applying Lemma 2 we deduce that E_β is equivalent to the pull-back extension $E\mathcal{I}_\beta$. Then E_β splits. By Proposition 1 for every $\alpha < \omega$ there exist a continuous homomorphism $R_\alpha : G_\alpha \rightarrow \pi^{-1}(G_\alpha)$ with $\pi \circ R_\alpha = \text{Id}_{G_\alpha}$. Consider the homomorphism

$$R : \begin{array}{ccc} \bigoplus_{\alpha < \omega} G_\alpha & \longrightarrow & X \\ (g_\alpha)_{\alpha < \omega} & \longmapsto & \sum_{\alpha < \omega} R_\alpha(g_\alpha) \end{array}$$

Using the universal property of the coproduct topology we deduce that R is continuous. Since

$$\pi\left(R((g_\alpha)_{\alpha < \omega})\right) = \pi\left(\sum_{\alpha < \omega} R_\alpha(g_\alpha)\right) = \sum_{\alpha < \omega} \pi(R_\alpha(g_\alpha)) = (g_\alpha)_{\alpha < \omega}$$

we obtain that $\pi \circ R = \text{Id}_{\bigoplus_{\alpha < \omega} G_\alpha}$ and E splits.

Let us check that ϕ is onto. Pick a family of extensions of topological abelian groups $\{\mathcal{E}_\alpha : 0 \rightarrow H \xrightarrow{\iota_\alpha} X_\alpha \xrightarrow{\pi_\alpha} G_\alpha \rightarrow 0 : \alpha < \omega\}$. From Lemma 21 we deduce that the sequence

$$\bigoplus_{\alpha < \omega} \mathcal{E}_\alpha : 0 \longrightarrow \bigoplus_{\alpha < \omega} H \xrightarrow{\bigoplus_{\alpha < \omega} \iota_\alpha} \bigoplus_{\alpha < \omega} X_\alpha \xrightarrow{\bigoplus_{\alpha < \omega} \pi_\alpha} \bigoplus_{\alpha < \omega} G_\alpha \longrightarrow 0$$

is an extension of topological abelian groups. Define the continuous homomorphism

$$P : \begin{array}{ccc} \bigoplus_{\alpha < \omega} H & \longrightarrow & H \\ (h_\alpha)_{\alpha < \omega} & \longmapsto & \sum_{\alpha < \omega} h_\alpha \end{array}$$

and take the push-out extension $P \bigoplus_{\alpha < \omega} \mathcal{E}_\alpha$ as in Lemma 3. For every $\beta < \omega$ the following diagram is commutative

$$\begin{array}{ccccccc} \mathcal{E}_\beta : & 0 & \longrightarrow & H & \xrightarrow{\iota_\beta} & X_\beta & \xrightarrow{\pi_\beta} G_\beta \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \downarrow \\ \bigoplus_{\alpha < \omega} \mathcal{E}_\alpha : & 0 & \longrightarrow & \bigoplus_{\alpha < \omega} H & \xrightarrow{\bigoplus \iota_\alpha} & \bigoplus_{\alpha < \omega} X_\alpha & \xrightarrow{\bigoplus \pi_\alpha} \bigoplus_{\alpha < \omega} G_\alpha \longrightarrow 0 \\ & & & \downarrow P & & \downarrow & \parallel \\ P \bigoplus_{\alpha < \omega} \mathcal{E}_\alpha : & 0 & \longrightarrow & H & \longrightarrow & PO & \longrightarrow \bigoplus_{\alpha < \omega} G_\alpha \longrightarrow 0 \end{array} \quad (5)$$

From Lemma 2 and the commutativity of (5) follows that \mathcal{E}_β is equivalent to $(P \bigoplus_{\alpha < \omega} \mathcal{E}_\alpha)\mathcal{I}_\beta$ for every $\beta < \omega$ and therefore $\phi([P \bigoplus_{\alpha < \omega} \mathcal{E}_\alpha]) = ([\mathcal{E}_\beta])_{\beta < \omega}$ \square

Remark 23. It would be interesting to find out if Theorem 22 is true for uncountable coproducts of topological abelian groups. Unfortunately, to prove Lemma 21 it is necessary to use that for every countable family of topological abelian groups $\{G_\alpha : \alpha < \omega\}$, the coproduct topology on the direct sum $\bigoplus_{\alpha < \omega} G_\alpha$ coincides with the box topology.

It is worth mentioning that Fulp and Griffith proved in [12, Theorem 2.13] that if $\{G_\alpha : \alpha < \kappa\}$ is a family of groups in \mathcal{L} such that $(\bigoplus_{\alpha < \kappa} G_\alpha, \tau_{\text{box}}) \in \mathcal{L}$ then $\text{Ext}((\bigoplus_{\alpha < \kappa} G_\alpha, \tau_{\text{box}}), H) \cong \prod_{\alpha < \kappa} \text{Ext}(G_\alpha, H)$ where $H \in \mathcal{L}$ and τ_{box} is the box topology.

5 Relations between Ext and Ext_{TVS}

Theorem 24. *Let $E : 0 \rightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \rightarrow 0$ be an extension of topological abelian groups. Suppose that Y is a Fréchet topological vector space and Z is a metrizable topological vector space. Then X admits a compatible topological vector space structure in such a way that E becomes an extension of (metrizable) topological vector spaces.*

Proof. Metrizable is a three space properties (see [13, 5.38(e)]) hence X is metrizable.

Since Y is metrizable and complete, it is Čech-complete and we can apply Lemma 5 to deduce that the completion $\varrho E : 0 \rightarrow Y \xrightarrow{\varrho i} \varrho X \xrightarrow{\varrho \pi} \varrho Z \rightarrow 0$ is an extension of topological abelian groups. According to [7, Proposition 2] there exist a compatible topological vector space structure in ϱX and we can regard ϱE as an extension of topological vector spaces. Consider the canonical inclusion $\mathcal{I} : Z \rightarrow \varrho Z$. The following diagram is commutative

$$\begin{array}{ccccccc} \varrho E : & 0 & \longrightarrow & Y & \xrightarrow{\varrho i} & \varrho X & \xrightarrow{\varrho \pi} \varrho Z \longrightarrow 0 \\ & & & \parallel & & \uparrow & \uparrow \mathcal{I} \\ E : & 0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{\pi} Z \longrightarrow 0 \end{array}$$

Since ϱE is an extension of topological vector spaces and \mathcal{I} is a continuous linear mapping, in virtue of Lemma 2, E is equivalent to the pull-back extension of topological vector spaces $(\varrho E)\mathcal{I} : 0 \rightarrow Y \rightarrow PB \rightarrow Z \rightarrow 0$. Let $T : X \rightarrow PB$ be the topological isomorphism that witnesses the equivalence of E and $(\varrho E)\mathcal{I}$. Take in X the topological vector space structure induced by T . This completes the proof. \square

Theorem 25. *Let $E : 0 \rightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \rightarrow 0$ be an extension of topological abelian groups. Suppose that Y is a complete locally bounded topological vector space and Z is a locally bounded topological vector space. Then X admits a topological vector space structure in such a way that E becomes an extension of (locally bounded) topological vector spaces.*

Proof. X is locally bounded because local boundedness is a three space property (see [18, Theorem 3.2]). Y is metrizable because every locally bounded Hausdorff topological vector space is metrizable. Hence Y is in particular almost metrizable and we can apply Lemma 5 to deduce that the completion $\varrho E : 0 \rightarrow \varrho Y \xrightarrow{\varrho i} \varrho X \xrightarrow{\varrho \pi} \varrho Z \rightarrow 0$ is an extension of topological abelian groups. Since Z is locally bounded, its completion ϱZ is also locally bounded and we are in the conditions of [5, Theorem 4]. So ϱE can be regarded as an extension of topological vector spaces. From here proceed as in the proof of Theorem 24. \square

Remark 26. Notice that in theorems 24 and 25, when we construct the topological vector space structure on X , the inclusion $X \hookrightarrow \varrho X$ turns out to be a linear map. Consequently, the multiplication by scalars on X , say $*_X : \mathbb{R} \times X \rightarrow X$, can be regarded as the restriction of the multiplication by scalars on ϱX , say $*_{\varrho X} : \mathbb{R} \times \varrho X \rightarrow \varrho X$. Completing the topological vector space $(X, *_X)$ we obtain another topological vector space $(\varrho X, \varrho(*_X))$. Thus the multiplication by scalars $\varrho(*_X) : \mathbb{R} \times \varrho X \rightarrow \varrho X$ coincides

with $*_X$ in $\mathbb{R} \times X$. Finally, since the continuous homomorphism $\varrho(*_X)$ coincides with the continuous homomorphism $*_{\varrho X}$ in the dense subgroup $\mathbb{R} \times X$, we conclude that both operations are the same.

Corollary 27. *Let Y and Z be topological vector spaces.*

- (i) *If Y is Fréchet and Z is metrizable then $\text{Ext}(Z, Y) \cong \text{Ext}_{\text{TVS}}(Z, Y)$.*
- (ii) *If Y is complete locally bounded and Z is locally bounded then $\text{Ext}(Z, Y) \cong \text{Ext}_{\text{TVS}}(Z, Y)$*

Corollary 28. *Let X be an abelian topological group and let $\pi : X \rightarrow Z$ be an open continuous homomorphism of X onto a topological vector space Z .*

- (i) *If $\ker \pi$ is a Fréchet space and Z is metrizable then X is a (metrizable) topological vector space.*
- (ii) *If $\ker \pi$ is a complete locally bounded topological vector space and Z is locally bounded then X is a (locally bounded) topological vector space.*

Corollary 29. *A metrizable topological vector space Z satisfies $\text{Ext}(Z, \mathbb{R}) = 0$ if and only if ϱZ is a \mathcal{K} -space.*

Acknowledgements

The author gratefully acknowledges the many helpful suggestions of María Jesús Chasco, Xabier Domínguez and Mikhail Tkachenko during the preparation of the paper. The author thanks the referee as well for the careful report. Finally, the author wishes to thank the Asociación de Amigos de la Universidad de Navarra and the Spanish Ministerio de Economía y Competitividad (grant MTM 2013-42486-P) for their financial support.

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